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Philosophy of Mathematical Practice: A Primer for Mathematics Educators

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Abstract

In recent years, philosophical work directly concerned with the practice of mathematics has intensified, giving rise to a movement known as the *philosophy of mathematical practice*. In this paper we offer a survey of this movement aimed at mathematics educators. We first describe the core questions philosophers of mathematical practice investigate as well as the philosophical methods they use to tackle them. We then provide a selective overview of work in the philosophy of mathematical practice covering topics including the distinction between formal and informal proofs, visualization and artefacts, mathematical explanation and understanding, value judgments, and mathematical design. We conclude with some remarks on the potential connections between the philosophy of mathematical practice and mathematics education.

1. Introduction

A wide variety of fields are concerned with the study of mathematics as a human practice, including anthropology, history, pedagogy, psychology, and sociology. Many philosophers have also engaged deeply with the mathematical practices of their time, from Plato, Descartes, Leibniz, Berkeley, and Kant to Frege, Russell, Hilbert, and Wittgenstein. In recent years, philosophical work directly concerned with the practice of mathematics has intensified, giving rise to a movement known as the *philosophy of mathematical practice*. If one accepts the premise of this special issue—that mathematicians' practice matters to mathematical instruction—then work in the philosophy of mathematical practice is potentially relevant to mathematics education research. To foster interactions between philosophical and pedagogical approaches to the study of mathematical practice, we provide in this paper an overview of the philosophy of mathematical practice by (i) highlighting its driving questions and methods and (ii) surveying some of its recent developments.

Perhaps the first steps towards a philosophy of mathematical practice were taken by Wilder (1950, 1981), who argued that mathematics is a cultural system, and Pólya (1945, 1954, 1962) who investigated mathematical problem solving, heuristics, and discovery. Lakatos (1976) dedicated his famous dialogue *Proofs and Refutations* to Pólya, as well as to Popper, the philosopher of science. In this work, he argued that mathematical knowledge grows not via the continuous production of formal derivations but by a dynamic process which involves proposing "proofs" which are then refuted and subsequently refined. Kitcher (1984) was also concerned with the growth of mathematical knowledge, arguing that modern mathematics evolved via a series of rational transitions from earlier practices.

In the past twenty years, contributions to the philosophy of mathematical practice have considerably increased. One earlier work from this time period is that of Corfield (2003) who argues that philosophers should pay more attention to what mathematicians do and presents a variety of case studies that do exactly this. More recently, a number of important collected volumes and monographs have been published that document the field. The collections edited by Van Kerkhove and Van Bendegem (2007), Van Kerkhove (2009), and Löwe and Müller (2010) bring together a variety of contributions focusing on the question of what a mathematical practice is and how it should be studied. The collection edited by Mancosu (2008) presents contemporary work on topics including visualization and diagrams, explanation and understanding, and mathematical concepts and definitions, among others. The collection edited by Ferreirós and Gray (2006) aims to promote connections between the history and philosophy of mathematics while the collection edited by Larvor (2016a) focuses on mathematical culture. Giaquinto (2007) provides a wide-ranging epistemological study of visual thinking in mathematical practice which takes into account empirical results from cognitive science and mathematics education. Grosholz (2007) has investigated what she calls the "productive ambiguity" of representations in mathematical and scientific practices through a diverse collection of case studies. Macbeth (2014) has given center stage to analyses of various past and contemporary mathematical practices in her general account of reason as a power of knowing. Ferreirós (2015) has offered an account of mathematical knowledge highlighting the importance of interactions between different practices. Finally, Wagner (2017) has proposed a philosophy of mathematical practice grounded in the negotiation of constraints.

In the remainder of this paper we focus primarily on contemporary philosophy of mathematical practice, with the aim of providing an overview of the field for mathematics education researchers.[1] We begin in sections (2) and (3) with a discussion of the questions investigated by philosophers of mathematical practice and the methods used to tackle them, respectively. Section (4) then surveys a selection of work in the philosophy of mathematical practice. Section (4.1) discusses the relationship between formal and informal proofs. Visualization and artefacts are the focus of section (4.2). Explanation and understanding are treated in section (4.3) while value judgments more generally are considered in section (4.4). Section (4.5) discusses the issue of design in mathematics. We conclude in section (5) with some remarks on the relation between the philosophy of mathematical practice and mathematics education.

[1] For other overviews of the field aimed at philosophers we refer to Van Bendegem (2014), Giardino (2017b), and Carter (2019).

2. Questions in the Philosophy of Mathematical Practice

Philosophy of mathematics, broadly construed, aims to understand mathematics, and potentially engage with how mathematics should be done. However, philosophers of mathematics of the twentieth century have focused mainly on the foundations of mathematics and on metaphysical and epistemological questions about the nature of numbers and our access to them (for a review, see Benacerraf and Putnam 1964; Shapiro 2000). Addressing these questions did not require paying close attention to mathematical practice beyond set theory and the elementary parts of arithmetic and geometry (Mancosu 2008, pp. 1–2). Philosophers of mathematical practice aim to broaden this research agenda and to engage directly with mathematics as it is practiced by addressing questions such as:

- What are the components of mathematical knowledge?
- What do *we* do when we do mathematics?
- What is "good" mathematics?
- In what sense is mathematics a social practice?
- What can the history of mathematics tell us about its nature?
- What is the relationship between mathematics and other disciplines?

Let us say a little about each question in turn.

What are the components of mathematical knowledge? Mathematical knowledge does not lie solely in theorems. Philosophers of mathematical practice investigate all the components of mathematical knowledge including proofs (e.g., Rav 1999; Detlefsen 2009), axioms (e.g., Schlimm 2013), concepts and definitions (e.g., Tappenden 2008), and methods (e.g., Avigad 2006). The objective is to analyze the nature of these different components and to understand the way(s) they function in actual mathematical practice.

What do *we* do when we do mathematics? Mathematics is done by *human* agents. Philosophers of mathematical practice take this seriously and so investigate how we, as human beings, do mathematics. In this respect, topics that have received philosophical attention include the role of visualization (e.g., Giaquinto 2007) and imagination (e.g., Arana 2016), the use of artefacts such as diagrams (e.g., Giardino 2017a) and notations (e.g., Muntersbjorn 1999; De Cruz and De Smedt 2013), and the importance of planning agency in mathematical activities (Hamami and Morris, forthcoming). Insofar as human agents are cognitive agents, several authors investigate and integrate this cognitive dimension, often by drawing upon empirical research from cognitive science and mathematics education (e.g., Giaquinto 2007).

What is "good" mathematics? Mathematicians want their work to be more than just correct—they also want it to be "good." Philosophers of mathematical practice aim to clarify the ways in which a piece of mathematics can be "good," for example, by being explanatory (e.g., Steiner 1978), beautiful (e.g., Rota 1997), pure (e.g., Detlefsen and Arana 2011), deep (e.g., Arana 2015), fruitful (e.g., Tappenden 2012), fitting (e.g., Raman-Sundström and Öhman 2018), interesting (e.g., Thomas 2017) or well-motivated (e.g., Morris 2020). Moreover, philosophers of mathematical practice are interested in good mathematical design. That is to say, they are interested in determining how we should evaluate mathematical notations and how mathematics should be structured (see, e.g., Sieg 2010; De Toffoli 2017; Avigad, 2020).

In what sense is mathematics a social practice? Mathematical practice is inherently social. Mathematicians submit their work for peer review, read journal articles, attend conferences, and work together to prove theorems. Philosophers of mathematical practice aim to understand this social dimension of mathematics (Löwe and Müller 2010; Ferreirós 2015; Larvor 2016a) by investigating its social values, norms, and practices (see, e.g., Müller-Hill 2009; Geist, Löwe, and Van Kerkhove 2010; Andersen, forthcoming; Rittberg, Tanswell, and Van Bendegem, forthcoming).

What can the history of mathematics tell us about its nature? Mathematics is not static—its history shows that it has undergone striking changes. Philosophers of mathematical practice thus undertake historical case studies to better understand these changes and the consequences associated with them (see, e.g., Manders 2008; Yap 2011; Avigad and Morris 2014, 2016; Ferreirós 2015). Indeed, many researchers in the field combine philosophical and historical expertise and aim to promote an integrated approach to the philosophy and history of mathematical practice (Mancosu, Jørgensen, and Pedersen 2005; Ferreirós and Gray 2006; Mumma and Panza 2012).

What is the relationship between mathematics and other disciplines? It is a truism to say that mathematics is used in virtually all branches of the social and natural sciences. Philosophers of mathematical practice are interested in investigating the various relationships between mathematics and other disciplines such as, e.g., computer science (Avigad 2008a, 2008b), physics (Urquhart 2008b, 2008a), and biology (Islami and Longo 2017), and to analyze what this tells us about mathematics. Furthermore, the traditional philosophical issue of the applicability of mathematics to the external world remains a central issue in the field (Wilson 2006; Pincock 2012).

In the next section, we discuss the methodologies employed by philosophers of mathematical practice to make progress on these questions. In section (4), we provide illustrations of their work. Our focus in this paper is on work that addresses the first three questions.

3. Methodologies in the Philosophy of Mathematical Practice

A distinctive characteristic of the philosophy of mathematical practice, inasmuch as it instantiates a *philosophical* approach to the study of mathematical practice, is of course the use of traditional methodologies in philosophical research. But armchair philosophy—as consisting of pure *a priori* thinking without any recourse to observation or experiment—does not seem appropriate to investigate mathematical practice. Rather, if the promise to pay greater attention to actual mathematical practice is to be fulfilled, one needs specific methods to look at and investigate mathematical practice. For this reason, the method of case studies has played a central role in the development of the field, and the use of various empirical methods borrowed from other fields is becoming more and more prominent. In this section, we provide an overview of the main methodologies that have been used so far in the philosophy of mathematical practice.

Case studies As will become clear in the following sections, the method of *case studies* is central to the philosophy of mathematical practice. The method consists in analyzing in detail one or more phenomena in the context of a specific mathematical practice, either past or present. Although the method is often used in an exploratory way to identify interesting phenomena, in published work it is almost always directed towards one or more special issues to be addressed. In particular, case studies can be used to support, or to provide a counterexample to, a given conceptual analysis or framework. As we shall see, the method has been used in relation to most of the issues we shall touch upon in our overview of the field (section (4)).

Conceptual analysis One of the central methods of philosophy is *conceptual analysis*. It consists in answering questions of the form "What is X?" by reflecting on our own understanding of the concept X, the result of which most often takes the form of a set of necessary and sufficient conditions for something to be or to count as X. Typical examples of philosophical concepts that have been analyzed in this way are agency, justice, knowledge, morality, rationality, and truth, to mention a few. In the philosophy of mathematical practice, the method has been used, in particular, to analyze different values attributed by mathematicians to pieces of mathematics such as proofs, theorems, definitions, or methods. One can then find in the literature conceptual analyses of what it means for a piece of mathematics to be or to count as explanatory, beautiful, pure, deep, fruitful, fitting, or interesting (see section (4.4)).

Conceptual framework Many phenomena and concepts do not lend themselves to conceptual analysis but require instead the development of a whole *conceptual framework*, that is, a complex network of notions and theses. Perhaps the most illustrative examples of this are conceptual frameworks aiming to provide an account of the notion of mathematical practice itself such as the ones proposed by Kitcher (1984), Van Bendegem and Van Kerkhove (2004), and Ferreirós (2015) which, of course, requires much more than a set of necessary and sufficient conditions.[2]

Arguments Developing *arguments* in favor of or against particular philosophical views is another fundamental method of philosophical research. This method has been central to traditional philosophy of mathematics in the twentieth century, where much effort has been devoted to the articulation of various philosophical views—naturalism, nominalism, Platonism, structuralism, etc.—which has generated in turn a whole dialectic of arguments

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[2] For discussions relevant to the characterization of the notion of mathematical practice, see also Giardino et al. (2012) and Carter (2019, sec. 4).

advanced in favor of and against them. Philosophical views allowing such a dialectic have still to emerge in the philosophy of mathematical practice. But arguments have been developed in the field with respect to specific theses and accounts. We can mention here the arguments put forward by Brown (1997, 1999) and Giaquinto (2007, 2011) in favor of the thesis that pictures and diagrams can play a positive epistemic role in inference and proof, by Lange (2009) against the thesis that proofs by mathematical induction are explanatory, and by Tanswell (2015) against the account of informal proofs proposed by Azzouni (2004), among others. In line with traditional philosophical practice, arguments in favor of a particular view or thesis can consist in showing that (i) it can account for a wide range of cases, (ii) it is in line with mathematical practice, and/or (iii) it relies on a minimal set of assumptions. Arguments against a view or thesis can take the form of (i) providing counterexamples, (ii) deriving unwanted consequences, and/or (iii) exposing problematic assumptions.

Empirical methods There is now a growing interest in using *empirical methods* in the philosophy of mathematical practice (Aberdein and Inglis 2019). These methods are usually borrowed from other fields such as the social sciences, cognitive science, and mathematics education, and research using these methods is often conducted in interdisciplinary collaborations with researchers of these different fields. So far the main empirical methods that have been used in the field are interviews, psychological experiments, and surveys. They have been employed to investigate the peer review process in mathematical practice (Geist, Löwe, and Van Kerkhove 2010; Andersen, forthcoming), the way mathematicians write mathematical research papers (Andersen, Johansen, and Sørensen, forthcoming) and evaluate the value(s) of mathematical proofs (Inglis and Aberdein 2014), knowledge ascriptions on the basis of mathematical proofs (Müller-Hill 2009), the kind of explanations mathematicians provide in online mathematical discussions (Pease, Aberdein, and Martin 2019), and the cognitive bases of Euclidean diagrammatic reasoning (Hamami, Mumma, and Amalric, submitted), among others.

Imports from other fields Finally, some contributions have imported and built upon developments from other fields. An illustrative example of this is the work of Avigad on value judgments (Avigad 2006) and mathematical understanding (Avigad 2008b) where he relied on formal models from the field of formal verification, as well as his more recent work which imports the notion of modularity from software engineering to analyze the structure and components of mathematical knowledge (Avigad, 2020). Another important example is to be found in the work of Giaquinto (2007) which drew heavily on works from cognitive science and mathematics education in developing his epistemological analysis of visual thinking in mathematics.

The philosophy of mathematical practice thus uses a wide range of philosophical methods and borrows methods from other fields as well. Research in the field has so far been pursued in a multidisciplinary and open-minded spirit, and so it is very likely that the number of methods employed in the philosophical study of mathematical practice will continue to grow with the development of the field.

4. A Selective Overview of Current Trends and Issues

We provide in this section an overview of some of the main trends and issues driving the philosophy of mathematical practice. Such an overview is bound to be selective but it offers nonetheless a representative picture of the type of research conducted in the field.[3]

4.1. Informal vs Formal Proofs

For the vast majority of the twentieth century, the dominant philosophical conception of what a proof is has been given by the notion of *formal proof* coming from logic and the foundations of mathematics. A formal proof is always specified relative to a given *formal deductive system*, and so defining the former requires first defining the latter. A formal deductive system Γ is a triple $\langle L, R, A \rangle$ composed of: a *formal language* L for representing mathematical propositions as formulas; a set of *rules of inference* R specifying what counts as a legitimate or acceptable inference from some formulas taken as premisses to another formula taken as conclusion; and a set of *axioms* A consisting of certain formulas in L. A formal proof in Γ is then defined as a sequence of formulas in L such that each formula in the sequence is either an axiom in A or the result of the application of a rule of inference in R to one or more preceding formulas in the sequence. The notion of formal proof constitutes then a *normative* account of what a mathematical proof *ought* to be, but it has also been taken as a *descriptive* account of what mathematical proofs are in practice.

Although many had noticed the limits of the model of formal proof as a descriptive account of ordinary proofs, Rav's essay entitled "Why do we prove theorems?" (Rav 1999) has been instrumental in triggering a philosophical discussion of the specificities of proofs in practice as well as of the relation between ordinary proofs and formal proofs. In this contribution, Rav has insisted on three features of informal proofs that are not—and maybe even cannot be—captured by the model of formal proof. The first one is what Rav calls the "irreducible semantic content" (Rav 1999, p. 11) of ordinary proofs which is precisely what is lost when an informal proof is translated into a formal proof—the latter being, by definition, a *syntactic* entity. The second one is the catalyzing role that the search for informal proofs plays in the growth of mathematical knowledge, that is, in the development of new mathematical tools, methods, concepts, etc. The third one is the capacity of informal proofs to serve as vehicles for various forms of *practical knowledge* or *know-how*, as Rav put it: "The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, the establishment of interconnections between theories, the systematisation of results—the entire mathematical know-how is embedded in proofs" (Rav 1999, p. 20).[4] According to Rav, the mathematical knowledge embedded in ordinary

[3] Two themes that will not be addressed in this paper are the role of computers in mathematical inquiry and the issue of ethics in mathematics. For the former, we refer the reader to Avigad (2008a). For the latter, see Rogers and Kaiser (1995), Ernest (2016, 2018), and Rittberg, Tanswell, and Van Bendegem (2018).

[4] For a discussion of Rav's notion of know-how, see Tanswell (2016, chapter 4).

proofs is of greater epistemological significance than the one embedded into the theorems they establish, and thus deserves dedicated epistemological attention.

The differences[5] between informal proofs and formal proofs have motivated several authors to develop new philosophical accounts of the former, the main objective being to provide a conception of mathematical proofs that is more faithful to the reality of mathematical practice. In this respect, Aberdein (2006) has proposed to analyse informal proofs using the resources of *informal logic*—a domain which aims to investigate inferential and argumentative practices in concrete or "real life" instances. Aberdein's study has emphasized the importance of taking into account the dialogical context in which informal proofs occur—following the theory of Walton (1998)—and has identified different types of "proof dialogues" in which informal proofs function. Another proposal is due to Leitgeb (2009) who has developed an account in which informal proofs differ from formal proofs in possessing *semantic* and *intuitive* components. In Leitgeb's analysis, the semantic dimension refers to the fact that the terms and sentences occurring in informal proofs possess a *meaning* and must be *interpreted*, while the intuitive dimension refers to the fact that the elementary steps of informal proofs, as well as the axioms adopted in practice, are taken to be intuitive in specific senses of the term. Starting from the observation that informal proofs "suffer some sort of violence or essential loss" (Larvor 2012, p. 717) when recast or translated into formal proofs, Larvor (2012) has developed an account of informal proofs that could explain this observation. Larvor's account is based on two key ideas: (1) that the validity of inferences in informal proofs does not depend only on the logical form of premisses and conclusions but also on their *content*; (2) that inferences in informal proofs should be conceived as *inferential actions* which can act not only on propositions but also on non-propositional representations such as "diagrams, notational expressions, physical models, mental models and computer models" (Larvor 2012, p. 721). Insofar as most of these inferential actions are dependent on the particular mathematical domain to which they belong they are not formal, and so are essentially resistant to any formal translation—the inferential actions occurring in formal proofs being by definition possible and acceptable in all domains.

Among the issues related to the nature of informal proofs that have attracted the attention of philosophers of mathematical practice is the important one of the *rigor* of informal proofs. Burgess (2015) provides a neat formulation of the notion of rigor to be investigated: "The quality whose presence in a purported proof makes it a genuine proof by present-day journal standards, and whose absence makes the proof spurious in a way that if discovered will call for retraction, is called *rigor*" (Burgess 2015, p. 2). But what does it mean to say that a mathematical proof is rigorous? A common answer to this question, that we might call the *standard view*, has it that a mathematical proof is *rigorous* whenever it can be *routinely translated* into a formal proof (Mac Lane 1986, p. 377). This conception of the rigor of mathematical proofs has, however, been heavily attacked in the philosophical literature, mostly on the ground that it yields an implausible account of how the notion of rigor is used by mathematicians to judge proofs in mathematical practice (Antonutti

[5] See Hamami (2018) for an analysis of these differences at the level of the elementary components of the two types of proofs.

Marfori 2010; Detlefsen 2009; Tanswell 2015; Larvor 2016b). A current objective for the philosophy of mathematical practice is thus to provide an account of the rigor of mathematical proofs that accords with the way proofs are judged to be rigorous in ordinary mathematical practice.[6]

To sum up, a key challenge for the philosophy of mathematical practice is to better understand the relation between formal and informal proofs and to work towards a philosophical account of mathematical proofs able to get at the subtle and multidimensional functioning of proofs in mathematical practice.

4.2. Visualization and Artefacts

In twentieth century philosophy of mathematics, little attention has been devoted to the nature and role of non-linguistic representations in mathematics. This may be explained, in part, by the well-known dismissal of intuition and visualization in mathematical thinking originating in nineteenth century mathematics (Pasch 1882; Hahn, 1933/1980), following mathematical developments such as the emergence of non-Euclidean geometries and the arithmetization of analysis. Yet, it must be acknowledged that visual representations are widespread in both past and contemporary mathematical practices. One of the most active lines of research within the philosophy of mathematical practice has thus consisted in investigating the role and nature of visual representations in various mathematical activities—proving, justifying, discovering, explaining, understanding, etc. This encompasses *internal* visual representations relying on visual imagery or imagination but also *external* representations—e.g., diagrams, symbolic systems, notations, computer images, etc.—which we will refer to as mathematical *artefacts*. As representative of this line of research we can mention the two monographs by Brown (1999) and Giaquinto (2007), the collected volume edited by Mancosu, Jørgensen, and Pedersen (2005), and the special issue of *Synthese* edited by Mumma and Panza (2012) on the history and philosophy of diagrams in mathematics. In this section, we propose a brief, and *a fortiori* selective, overview of these developments.[7]

In opposition to the received view, Brown (1997, 1999) and Giaquinto (2007, 2011) have both defended a positive epistemic role for visual representations in mathematics thereby triggering renewed interest in this topic. Brown (1997, 1999) has argued that pictures can prove mathematical propositions—the third chapter of the book presents and discusses a wealth of examples of "picture-proofs," ranging from the intermediate value theorem to various propositions about natural numbers and infinite series.[8] Brown's view on the role of pictures is integrated with his defense of Platonism, the main idea being that "*Some 'pictures' are not really pictures, but rather are windows to Plato's heaven*" (Brown 1999,

[6] See Hamami (forthcoming) for an attempt to meet this challenge which is nonetheless compatible with the standard view.

[7] For extensive reviews of visual and diagrammatic thinking in mathematics, see Giaquinto (2016) and Giardino (2017a).

[8] For a critical outlook on Brown's view, see Folina (1999).

p. 39), that is, pictures should be conceived as instruments helping us to perceive the mathematical realm. Giaquinto addresses in his book several themes related to the epistemology of visual thinking in mathematics, with a main focus on *discovery* which he defines as the process by which one can "come to believe [a truth] independently and in an epistemically acceptable way" (Giaquinto 2007, p. 2). The first part of the book makes a proposal about how we acquire basic geometric knowledge. Giaquinto argues that we possess certain belief-forming dispositions associated with our basic geometric concepts, drawing on contemporary works in cognitive science. Visual perception and imagination can trigger these dispositions, leading to the acquisition of geometric beliefs. When these belief-forming dispositions are reliable, the acquired beliefs count as knowledge, and they constitute then a case of synthetic *a priori* knowledge. The second part of the book examines the case of arithmetic, and the third part is concerned with the role of visual thinking in more advanced mathematics. An important point of the book is that visual thinking in mathematics is not uniform—there is a diversity of visual operations associated with different mathematical domains and contexts. In this respect, the last chapter of the book urges the development of a fine-grained taxonomy of the kinds of visual thinking present in mathematics, and in fact makes a substantial step forward in this direction.

The most common method to investigate visual representations in mathematical practice is that of *case studies*. A perfect representative of this approach is the seminal analysis by Manders (2008) of the functioning of Euclidean diagrams in the geometric proofs of Euclid's *Elements*. The modern view considers that Euclid's geometric proofs are flawed insofar as they contain deductive steps that rely essentially on the diagram such as the famous intersection point of the two circles in proposition 1 from book I of the *Elements*. In opposition to the modern critics, Manders aims to defend Euclid's diagram-based geometric practice by showing that diagrams are used in a highly controlled way in the context of Euclid's geometric proofs. More specifically, Manders provides an analysis of what he calls *diagram-based attributions*, that is, the claims introduced in the demonstration text which are based in totality or in part on the diagram. Central to his account is the distinction between the *exact* and *co-exact* attributes of Euclidean diagrams: a *co-exact* attribute is a condition which is insensitive to a certain range of continuous deformations of the diagram—e.g., that two circles intersect, or that an angle is contained in another one—an *exact* attribute is a condition that is sensitive to at least some of these deformations for which it obtains only in isolated cases—e.g., that a line is tangent to a circle, or that two segments are equal in length. The key insight of Manders is that diagram-based attributions in Euclid's geometric proofs are exclusively limited to co-exact attributes—exact attributes are never read off from the diagram, they must be established based on prior claims in the demonstration text. According to Manders, it is because co-exact attributes are sufficiently stable under the (concrete) drawing of (imperfect) diagrams that it can be controlled so as to ensure agreement among the members of the practice with respect to diagram-based attributions, thus explaining the incredible stability of the practice across centuries.[9]

[9] For another seminal analysis of the role of diagrams in Greek mathematics, see Netz (1999). For attempts to formalize diagrammatic reasoning in the proofs of Euclid's *Elements* see Miller (2007), Mumma (2006, 2010), and Avigad, Dean, and Mumma (2009). For an

More recently, the method of case studies has been applied to different kinds of visual representations used in contemporary mathematical practice. Carter (2010, 2012) has analyzed the role of diagrams in the production of a set of proofs in the field of free probability theory. On the basis of her case study, she has argued that diagrams provide a source of inspiration for identifying relevant definitions and proof strategies, that they can serve as "frameworks" for parts of a proof, and that they can help break a proof down into parts which are more manageable. Starikova (2010, 2012) has looked at the role of Cayley graphs in the development of geometric group theory. She has analyzed how the visual representations of Cayley graphs led to the discovery of new geometric properties of groups and to the introduction of new mathematical concepts, showing thereby that visual representations can play a substantial role in the development of new mathematics. De Toffoli and Giardino have investigated the use of diagrams in knot theory (De Toffoli and Giardino 2014) and low-dimensional topology (De Toffoli and Giardino 2015). In both cases, they have argued that the use of diagrams in these mathematical practices brings into play a certain form of *manipulative imagination* which is acquired by the practitioners through training and which allows them to perform *epistemic actions* (Kirsh and Maglio 1994) on the considered visual representations, thereby playing a substantial epistemic role in the contexts of proving and discovery. Finally, Eckes and Giardino (2018) have studied the role of diagrams in solving classification problems in the context of combinatorial topology—concerning the classification of compact surfaces—and algebra—concerning the classification of complex semisimple Lie algebras.

A topic that is receiving growing attention in the philosophy of mathematical practice is the one of *mathematical notations*. Brown (1999) and Colyvan (2012) have both dedicated a whole chapter of their introductory textbooks in the philosophy of mathematics to this topic. Brown (1999, chap. 6) presents various notations used in the contexts of number systems and knot theory and argues, from his Platonist position, that notations are a means to discover and reveal properties of mathematical objects, emphasizing also the importance of their *computational role*, that is, they make it possible to perform systematic computations. Colyvan (2012, chap. 8) reflects on what constitutes a good notation or notational system, and by working through diverse examples identifies several benefits that good notations possess such as enabling economical mathematical expressions, promoting generalizations and new mathematical developments, providing computational power, and enabling mathematical explanation and understanding. Macbeth (2012b) argues that good mathematical notations are not just "convenient shorthand" for mathematical expressions, they more importantly have the capacity to embody mathematical reasoning by formulating the content of mathematical concepts and making it possible to manipulate this content through a manipulation of the notation itself. Dutilh Novaes (2013) has defended the thesis that external symbolic systems play an essential role both in the acquisition of mathematical abilities by individuals as well as in the historical development of these abilities. Finally, De Cruz and De Smedt (2013) have put forward a view in which mathematical symbols are (1) constitutive of the mathematical concepts they represent and

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analysis of the relation between geometric objects and the diagrams that represent them, see Panza (2012).

(2) play a central role in mathematical cognition by supporting mathematical operations which constitute *epistemic actions* in the sense of Kirsh and Maglio (1994).

Significant progress has been made to understand the nature and role of visualization and artefacts in several areas of past and present mathematical practices. Efforts in this direction need to be pursued, but it will also be important to develop more unified frameworks and to provide a general conceptual toolbox for the study of specific cases.

4.3. Explanation and Understanding

Intuitively, some proofs establish *that* a theorem holds, while others show *why* it holds. Proofs of the latter kind are often called explanatory. Philosophers of mathematical practice have attempted to clarify what it means for a proof to be explanatory in more precise terms via conceptual analysis. Here we consider two well known accounts of explanation due to Steiner (1978) and Kitcher (1989). More recent accounts, which due to space constraints cannot be discussed here, include those by Frans and Weber (2014), Lange (2014) and Inglis and Mejía-Ramos (2019). We also briefly consider more recent developments that go beyond giving an analysis of what makes a proof explanatory, before discussing the connection between explanation and understanding.

In his 1978 paper, Steiner considers a variety of criteria for explanatory proofs, including generality and discoverability. While initially plausible, Steiner argues that these criteria do not capture mathematical explanation. To try to demonstrate this, he exhibits proofs which, he claims, satisfy the criteria but fail to be intuitively explanatory (Steiner 1978, pp. 130–40). Next Steiner presents his own account of mathematical explanation. On his account, an explanatory proof is one which depends on a "characterizing property" (Steiner 1978, p. 147) of something mentioned in the theorem. Moreover, it should be generalizable, so that if we change the characterizing property to that of a different, but related, mathematical object or structure then we see how the proof changes and thus arrive at a different theorem. Steiner then tests his account against the example proofs he used to rule out the discarded criteria of explanation and argues that his account judges these proofs correctly.

Steiner's account has attracted much discussion and criticism (see Resnik and Kushner 1987; Weber and Verhoeven 2002; Hafner and Mancosu 2005; Lange 2014; Pincock 2015). Many of the criticisms take the form of counterexamples developed via the method of case studies. That is to say, philosophers present a proof that they, or mathematicians, intuitively take to be explanatory (or non-explanatory), but which Steiner's account judges to be non-explanatory (or explanatory). For example, Hafner and Mancosu (2005) focus on a proof of Kummer's convergence test due to Pringsheim. Kummer's convergence test states that, for an arbitrary sequence of positive integers (B_n), if $\lim_{n\to\infty}(B_n \frac{a_n}{a_{n+1}} - B_{n+1}) > 0$ then the series $\sum a_n$ converges. The appearance of such an arbitrary sequence (B_n) in the test was puzzling to mathematicians, and Pringsheim explicitly intended his proof to explain its presence (Hafner and Mancosu 2005, p. 229). However, because the sequence is arbitrary, it has no characterizing property. This means Pringsheim's proof does not depend on a characterizing property and so fails to be explanatory on Steiner's account (Hafner and Mancosu 2005, p. 230). In other words, because Pringsheim's proof was designed to be explanatory, Steiner's account judges it incorrectly.

Kitcher's (1989) account of mathematical explanation is based on unification. It is rather technical, so what follows is only a rough description. For Kitcher, a proof in a mathematical theory K is explanatory if and only if it belongs to the "explanatory store" (Kitcher 1989, p. 430) of K. The explanatory store of K is the collection of proofs which most unifies K, where the degree to which a collection of proofs unifies a theory is based on the number of genuine and distinct derivation patterns or schemes its proofs instantiate and the number of theorems that are proven. The most unifying collection will do the best job at minimizing the number of derivation patterns instantiated while maximizing the number of theorems proven.

Philosophers including Hafner and Mancosu (2008), Lange (2014) and Pincock (2015) have offered counterexamples to Kitcher's account. For example, as many theorems can be proven via "a 'plug and chug' technique" (Lange 2014, p. 523), Lange argues that these "brute force" proofs will belong to a theory's explanatory store. They will thus be judged as explanatory on Kitcher's account. But such proofs are not intuitively explanatory, and so Kitcher's account judges them incorrectly. Hafner and Mancosu's (2008) detailed analysis provides a concrete example of this. There is a decision procedure for an axiomatization of the theory of real closed fields and so all elementary results in this theory can be proved by using the same algorithm. All of these proofs therefore instantiate the same derivation scheme. As all elementary results can be proven using just one argument scheme, the collection of these proofs constitutes the most unifying systematization of the theory of real closed fields, i.e., they constitute its explanatory store. This means that Kitcher's account judges them to be explanatory (Hafner and Mancosu 2008, pp. 165-166). However, such proofs are not intuitively explanatory. Indeed, Hafner and Mancosu note that the proofs may be so large that it is physically impossible to write them down (Hafner and Mancosu 2008, p. 159).

Further work on explanation has gone beyond giving an analysis of explanatory proofs. For example, D'Alessandro (forthcoming) argues that theorems can be explanatory and that a preoccupation with explanatory proofs has led to philosophical mistakes, while Lehet (forthcoming) argues that definitions can be explanatory. Finally Morris (forthcoming) has asked about the instrumental value of explanations in mathematics.

Closely related to mathematical explanation is the issue of *mathematical understanding* which has been the object of dedicated philosophical studies in recent years. One of the first articulated accounts of mathematical understanding is due to Avigad (2008b) who has proposed that an agent's understanding of a piece of mathematics—a proof, a theorem, a definition, a method, etc.—can be conceived in terms of the possession of certain mathematical abilities. From this perspective, providing an account of mathematical understanding in a specific case amounts to characterizing the relevant abilities, an approach that he has adopted with respect to the understanding of ordinary mathematical proofs. In this latter case, Macbeth (2012a) has argued that if we are to account for how mathematical proofs can convey mathematical understanding, we need a conception of mathematical proof different from the notion of formal proof, one that can explain how one can reason mathematically on the basis of content. Finally, Folina (2018) proposes that mathematical understanding encompasses a whole range of different phenomena and that it should be approached as a "family resemblance" concept. To this end, she has developed a structuralist perspective on mathematical understanding which is driven by the idea that mathematical structures are the subject matter of mathematics, and that the various components of mathematical understanding should be conceived as related to an understanding of its subject matter.

In sum, then, there has been much debate over the nature of explanation in mathematics, but there is as yet no widely accepted account of mathematical explanation. While understanding is closely related to explanation, it has only recently started to receive sustained philosophical investigation and so more work remains to be done in this direction.

4.4. Value Judgments

Mathematicians frequently judge pieces of mathematics on criteria other than correctness. For example, mathematicians may praise a theorem or a proof for its *beauty*, commend a proof for its *purity* or admire the *fruitfulness* of a mathematical method. Philosophers of mathematical practice aim to clarify these terms and make them more precise via conceptual analysis and the use of case studies. Here we briefly present the work of Rota (1997) and Cellucci (2015) on beauty, Detlefsen and Arana (2011) on purity, and Yap (2011) on fruitfulness.

Rota (1997) argues that when mathematicians judge a piece of mathematics to be beautiful they really mean that it is enlightening. He claims they speak of beauty instead of enlightenment because the latter concept has two features mathematicians dislike: it is hard to formally analyze and it is a fuzzy concept in the sense that a piece of mathematics can be partly enlightening (Rota 1997, p. 181). Rota does not offer a full analysis of enlightenment but suggests that it involves grasping the role of a mathematical statement, its relevance, and how it relates to other statements. He also insists that it is a logical, not just psychological, concept (Rota 1997, p. 181).

Cellucci (2015) argues that mathematical beauty consists in understanding rather than enlightenment, where understanding is to be analyzed in terms of grasping of fit. For example, on Cellucci's account, a proof is beautiful when it yields understanding, i.e., when it makes it clear how each of its parts fit with each other as well as with the proof as a whole (Cellucci 2015, sec. 10). As an example of a beautiful proof, Cellucci (2015, sec. 11) considers the problem from Plato's *Meno* of constructing a square with double the area of a given square. The diagram in figure (1) shows that while the area of the given square in the bottom left has an area equal to that of two right angled triangles, the area of the middle square is equal to four right angled triangles. A beautiful theorem, for Cellucci, yields understanding by establishing a clear relationship between fundamental mathematical concepts (Cellucci 2015, sec. 14). As an example of a beautiful theorem, he cites Euler's identity $e^{i\pi}+1=0$ (Cellucci 2015, sec. 14).

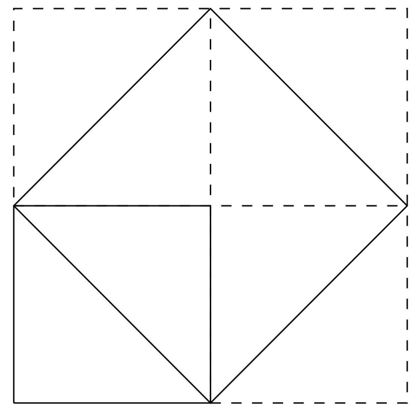

Fig. 1 The middle square has area twice that of the given square in the bottom left corner.

A preference for the use of “pure” methods in mathematics goes back as far as Aristotle and continues to the present day. Detlefsen and Arana (2011) aim to clarify what is meant by such references by offering a conceptual analysis of what they call topical purity and its epistemic benefits. They begin by defining the topic of a mathematical problem to be the set of mathematical resources including axioms, definitions, and inferences such that, if a mathematician retracted any one of these, then the content of the problem would change. They then call a solution to a problem topically pure when it only uses resources that determine the topic of the problem (Detlefsen and Arana 2011, p. 13).

As an example, consider the problem of determining whether there are infinitely many primes. The topic of this problem is determined by the axioms for successor, induction axioms, order axioms, and the definitions of prime, divisibility, and multiplication (Detlefsen and Arana 2011, p. 13). One solution to this problem, due to Furstenberg, uses topological resources, which are outside the topic of the problem since they can be retracted without changing the content of the problem (Detlefsen and Arana 2011, sec. 5). Furstenberg’s solution is thus impure.

Pure solutions have an important epistemic benefit: they are stable in a way that impure solutions are not. Detlefsen and Arana (2011, p. 13) point out that if a mathematician retracts any part of a topically pure solution then its content changes and the original problem dissolves. With a topically impure solution, however, there are parts that she can retract without dissolving the original problem. In other words, a topically pure solution will remain a solution for as long as the problem remains the same (Detlefsen and Arana 2011, p. 17).

In addition to wanting their proofs to be explanatory or beautiful and their solutions pure, mathematicians also want their concepts and calculi to be fruitful. Tappenden (2012) and Yap (2011) both offer accounts of what fruitfulness amounts to. Here we focus on Yap’s analysis, which begins with Gauss. In a letter to one of his students, Gauss suggested that a new mathematical system or calculus that does not provide any additional deductive power can still be fruitful if it “corresponds to the innermost nature of frequent wants” since “every one who assimilates it thoroughly” can solve problems “mechanically” (Yap 2011 Gauss quoted on pp. 410–411). Yap offers an interpretation of these phrases and thus fleshes out a way of characterizing fruitfulness which she illustrates with the case study of congruences.

Sometimes when writing proofs, we end up having to re-derive the same results again and again because we don't have a general theorem or lemma we can appeal to. Yap suggests that our frequent wants are the results that we have to re-derive, so a calculus that corresponds to them is one which provides us with theorems or lemmas that we can apply in our proofs to obtain those results without having to re-derive them (Yap 2011, p. 411). For example, important properties of congruences, such as preservation under multiplication, i.e., if $a \equiv b \pmod{c}$ and $a' \equiv b' \pmod{c}$ then $aa' \equiv bb' \pmod{c}$, are formulated as lemmas in congruence theory. If we use the theory of congruences, we can apply these lemmas directly in our proofs. If we don't use the theory of congruences and instead work directly with the notion of divisibility, then we can't apply these lemmas and must re-derive the corresponding results every time we want to use them. Consequently, proofs that use congruences are shorter and cleaner than those that do not (Yap 2011, p. 412).

Yap suggests that assimilating a new calculus thoroughly can be interpreted as adding its resources to existing methods. So, for example, part of assimilating congruences means adding their important properties, like preservation under multiplication, to the collection of results that we can use when proving theorems (Yap 2011, p. 412). Solving problems mechanically is interpreted by Yap as providing a systematic way of breaking them down into smaller problems which are easier to address. She uses Gauss's inductive proof of Quadratic Reciprocity as an example of a mechanical solution made possible by the use of congruences (Yap 2011, p. 414). Putting this all together, a fruitful calculus is thus one which allows us to systematically analyze problems into smaller, more manageable pieces by supplying us with tools that integrate with our existing methods and which allow us to avoid continually re-deriving the same results in our proofs.

Philosophers of mathematical practice have investigated a wide variety of value judgments in mathematics in addition to explanation, beauty, purity, and fruitfulness. For example Arana (2015) has explored depth, Raman-Sundström and Öhman (2018) have analyzed fit, Thomas (2017) has examined interestingness in mathematics and Morris (2020) has investigated proofs that are well-motivated. Philosophers are also interested in how these different values are related. For example, whether mathematical explanations are also beautiful and vice versa (see, e.g., Lange 2016).

In short, there are a wide variety of value judgments that mathematicians make about their mathematics. Philosophers aim to identify and develop precise accounts of these values so that they can be clarified and better understood.

4.5. Mathematical Design

Mathematicians often have to make design decisions in the course of their work. For example, a mathematician must choose which notation to use and how best to structure her proof. Here we present recent work by De Toffoli (2017) and Avigad (2020) that focuses on such design issues in mathematics.

De Toffoli (2017) identifies three criteria that can be used to evaluate a mathematical notation: expressiveness, calculability, and transparency. Expressiveness refers to the information that the notation captures. For example both Arabic and Roman numerals are equivalent with respect to expressiveness because they can be used to represent the same things (De Toffoli 2017, p. 165). Calculability refers to the calculations that the notation allows. For example, the Arabic numerals are superior to the Roman numerals with respect to their calculability, because they allow for much more efficient calculations (De Toffoli 2017, p. 165; see also Schlimm and Neth 2008). Finally transparency refers to how intuitive or natural the notation is to use. For example, in some ways the Roman numerals are more intuitive to use than the Arabic numerals because of their use of strokes to represent one, two, three, and four (De Toffoli 2017, p. 165).

Ideally we would like our notation to score highly on each of the criteria of expressiveness, calculability, and transparency, but De Toffoli notes that there are often trade-offs between them. She illustrates this with the examples of Euler and Venn diagrams (De Toffoli 2017, p. 164). Euler diagrams are a way of representing logical relationships visually and are very easy to use and understand. However, there are some cases they are unable to represent. They thus score highly on transparency, but less well on expressiveness. Venn diagrams were designed specifically to overcome some of the limitations of Euler diagrams by introducing additional conventions. While they thus score better on expressiveness, the additional conventions make them less intuitive to work with, and so they score lower on transparency.

De Toffoli does not provide an "algorithm" (De Toffoli 2017, p. 163) for evaluating a notation based on her criteria, as she argues that the context of the evaluation and goals of the notation must be taken into consideration. Generally, however, if two notations score just as well as each other on two of her criteria but one does better on the third, then the one that does better on the third is to be preferred.

Avigad (2020) considers not just the design of notations, but the design of mathematics in general. He argues that mathematics is designed to have a "modular" structure. Moreover, he argues that it should be so designed because modularity has a number of important benefits.

Avigad borrows the term "modular" from software engineering. He notes that from the 1960s onwards, software engineers began decomposing complex computer programs into a number of independent parts or "modules" (Avigad, 2020, sec. 3.2). Interactions between these modules and between the user and the modules happen via interfaces. These interfaces specify, for example, the input the module expects and the output it produces. Importantly, the nitty-gritty details about the workings of the code of the module are "hidden" or "encapsulated" (Avigad, 2020, sec. 3.4) and can be ignored. The complex computer program, when decomposed into independent parts whose interactions are limited and controlled, is an example of a system with a modular structure.

Avigad argues that mathematics is analogous to software engineering. He points out that we break down complex mathematical proofs into independent parts like sequences of definitions and lemmas. These definitions and lemmas serve to hide information. Consider, for example, the definition "x divides y" which says that there is a z such that $y = z \cdot x$. This definition hides information, in particular the value of z (Avigad, 2020, sec. 5.1). Similarly, when we break a proof down into a series of lemmas, the proofs of the lemmas are hidden from the body of the main proof, and the statement of the lemma serves as an interface (Avigad, 2020, sec. 4.3). We have thus designed our proofs to have a modular structure, much like modern computer programs.

Moreover, Avigad argues that mathematics *should* be modular because modularity brings numerous benefits. In particular, he argues that the benefits of modularity in software engineering carry over to mathematics and illustrates them with case studies from number theory (Avigad, 2020, sec. 5). These benefits include making it easier (Avigad, 2020, sec. 3.1): (i) to understand; (ii) to find problems or mistakes; (iii) for different agents to work on different pieces concurrently; (iv) to take components and use them again in a different context.

In sum, philosophers have conducted case studies and used work from other fields to help identify criteria for evaluating the design of mathematics and to assess the benefits and drawbacks of different ways of designing mathematics.

5. Conclusion

Our aim in this paper was to provide a survey of the philosophy of mathematical practice for mathematics educators by describing the main questions and methods in the field and illustrating them with key examples. We saw in section (2) that philosophers of mathematical practice tackle a wide variety of topics ranging from the various components of mathematical knowledge to the historical and social dimensions of mathematics. In section (3) we presented the main philosophical methods employed in the field, including traditional methods such as case studies and conceptual analysis, as well as more modern empirical ones. In section (4) we illustrated how these methods have been applied to tackle issues such as the relationship between formal and informal proofs, visualization and artefacts, explanation and understanding, value judgments, and mathematical design, thereby providing an overview of some of the current trends and issues driving the field.

As it turns out, mathematics education researchers are also concerned with many of the same issues. A quick look at the mathematics education literature attests to this. For instance, CadwalladerOlsker (2011) discusses formal and informal proofs and considers pedagogical issues that arise when teaching students how to write proofs. Presmeg (2006) addresses existing work on visualization in mathematics education and identifies a list of 13 important questions for future research in the area. Hanna (2018) contrasts proofs which are explanatory in a philosophical sense with proofs that are explanatory in an educational sense and argues that philosophical analyses may prove useful to mathematics education researchers. Sinclair (2004) argues that aesthetic experiences in mathematics can help motivate students. Erbas, Alacaci, and Bulut (2012) analyze the design of certain mathematical textbooks, considering, for example, how the mathematical content is structured. This indicates that there is a significant overlap between the issues addressed in the philosophy of mathematical practice and in mathematics education. So how can these two fields benefit from each other?[10]

On the one hand, the philosophy of mathematical practice can offer concepts, frameworks, and theories that can be used to frame, or be applied to, mathematics education research. For example, philosophers of mathematical practice have developed analyses of virtues like explanation and beauty, and have proposed accounts of how diagrams and notations are used in specific mathematical practices. Mathematics education researchers may want to teach students to recognize mathematical explanations or mathematical beauty, or to use certain diagrams and notations, and so may find these analyses to be a useful starting point for designing and testing educational policies.

On the other hand, mathematics education can offer philosophy of mathematical practice much needed empirical data as well as key insights into the transmission of mathematical knowledge. In most cases, philosophers of mathematical practice base their analyses on their own understanding of what is going on in mathematical practice, and at best appeal to the judgments of a small number of mathematicians. It would be much better if these analyses were constrained by concrete empirical data on mathematical practice, and as a matter of fact the mathematics education literature already provides a rich repertoire of empirical studies to borrow from. Furthermore, the transmission of mathematical knowledge through training is an essential part of mathematical practice, yet it has received little philosophical attention. Mathematics education can and ought to inform philosophical investigations in this direction.

There are thus many avenues for fruitful interactions and collaborations between the philosophy of mathematical practice and mathematics education. But as we mentioned at the very beginning, several other fields are also concerned with the study of mathematics as a human practice. Much is to be gained by recognizing that the study of mathematical practice is fundamentally a multidisciplinary enterprise, and that all the different fields involved can only benefit by interacting, communicating, and collaborating with each other.

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[10] Several important volumes have already brought together philosophers and mathematics educators to reflect on the relation between the two fields (François and Van Bendegem 2007; Van Kerkhove and Van Bendegem 2007; Ernest et al. 2016; Ernest 2018).

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